PHYS4150 — PLASMA PHYSICS

LECTURE 8 - MAGNETIC MIRRORS, ADIABATIC INVARIANTS

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1 MAGNETIC MOMENT AS A CONSTANT OF MOTION

We now investigate the guiding center motion of a charged particle along an inhomogeneous magnetic field. We assume that the field is axially symmetric (i.e. $\mathbf{B} = (B_\rho, B_\phi, B_z)$ with $\partial_\phi \mathbf{B} = 0$), where the symmetry axis z is aligned with the field gradient $\nabla \mathbf{B} = \partial_z B_z$. We only consider particle motions close to the symmetry axis where we can safely ignore the dependence of $\partial_z B_z$ on the radial distance ρ.

From Gauss' law $\nabla \cdot \mathbf{B} = 0$ follows that

$$\nabla \cdot \mathbf{B} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) + \frac{\partial B_z}{\partial z} = 0,$$

and after performing the integration with respect to ρ

$$B_\rho = -\frac{1}{2} \left(\frac{\partial B_z}{\partial z} \right) \rho. \tag{1}$$

Note that this relation is only valid close to the symmetry axis because we assumed that $\frac{\partial B_z}{\partial z} \neq f(\rho)$. The particle's motion parallel to the symmetry axis is given by

$$m \frac{\mathrm{d} v_z}{\mathrm{d} t} = F_z = q \left(v_x B_y - v_y B_x \right), \tag{2}$$

where the field components B_x and B_y are given by Eq. (1)

$$B_x = -\frac{1}{2} q \left(\frac{\partial B_z}{\partial z} \right) x, \tag{3}$$

$$B_y = -\frac{1}{2} q \left(\frac{\partial B_z}{\partial z} \right) y. \tag{4}$$

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With that we get

$$F_z = -\frac{1}{2} q \frac{\partial B_z}{\partial z} \left(v_x y - v_y x \right). \tag{5}$$

Let us now assume that $\frac{\partial B_z}{\partial z}$ is small, so that the motion in the x-y plane will be circular

$$x = \rho_c \sin \omega_c t$$

$$y = \rho_c \cos \omega_c t \frac{q}{|q|}.$$

The $q/|q|$ term in the expression for y accounts for the direction of the gyro motion. The corresponding velocity components are then

$$v_x = \omega_c \rho_c \cos \omega_c t,$$

$$v_y = -\frac{q}{|q|} \omega_c \rho_c \sin \omega_c t,$$

and thus

$$F_z = -\frac{\partial B_z}{\partial z} \left(\frac{|q|}{2} \omega_c \rho_c^2 \right).$$

Recall that the *magnetic moment* is

$$\mu = \frac{m \mathbf{v}_\perp^2}{2B} = \frac{T_\perp}{B},$$

or after expressing $\mathbf{v}_\perp$ by ω_c and ρ_c

$\omega_c =; \rho_c =$

$$\mu = \left(\frac{|q|}{2} \omega_c \rho_c^2 \right),$$

and hence

$$\boxed{F_z = -\frac{\partial B_z}{\partial z} \mu.} \tag{6}$$

This result implies that the particle is repelled from strong magnetic field regions.

Now we have a closer look at the particle's azimuthal motion in the x-y plane. Here the force acting on the particle is

$$F_\phi = q v_z B_\rho, \tag{7}$$

from which follows that the rate of change of the kinetic energy of the motion in this plane is

$$\frac{\mathrm{d}T_\perp}{\mathrm{d}t} = v_\phi q v_z B_\rho.$$

After using Eq. (1) and replacing v_ϕ by $-q/|q|\mathbf{v}_\perp$ we find that

$$\frac{\mathrm{d}T_\perp}{\mathrm{d}t} = |q|\mathbf{v}_\perp v_z \frac{\partial B_z}{\partial z}\frac{\rho}{2}.$$

Note that while the total kinetic energy T is conserved, $T_\perp$ is not constant. After replacing ρ with ρ_c we get

$$\boxed{\dot{T}_\perp = \frac{T_\perp v_z}{B}\frac{\partial B_z}{\partial z}.} \tag{8}$$

Finally, knowledge of $\dot{T}_\perp$ enables us to derive the rate of change of the magnetic moment

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{T_\perp}{B}\right) = \frac{1}{B}\dot{T}_\perp - \frac{T_\perp}{B^2}\dot{B}.$$

Using that $\dot{B} = v_z\partial_z B_z$ yields

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{1}{B}\dot{T}_\perp - \frac{T_\perp}{B^2}v_z\frac{\partial B_z}{\partial z}$$

and after inserting Eq. (8)

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{T_\perp}{B^2}v_z\frac{\partial B_z}{\partial z} - \frac{T_\perp}{B^2}v_z\frac{\partial B_z}{\partial z} = 0.$$

The magnetic moment μ is a constant of motion for a $\mathbf{B}\|\nabla\mathbf{B}$ field configuration. Such a field configuration constitute a *magnetic mirror* – a particle moving into the strong field region will eventually reflected back into the weak field domain.

2 MAGNETIC MIRRORS CONFIGURATION AND THE CORRESPONDING ADIABATIC INVARIANTS

We have just found that the magnetic moment μ is a constant of motion for a magnetic mirror configuration. So far we ignored possible temporal changes of the field strength. However, we have learned in the previous lecture that for each cyclic degree of freedom exists one adiabatic invariant I as long as the temporal changes are slow. So the question is how many different cyclic motion does a particle perform in a magnetic mirror configuration and what are the corresponding adiabatic invariants?

2.1 *The gyro motion around the z-axis*

First we consider the motion in the x-y-plane and assume that the B_z component is constant. This is justified as long as the changes are small compared to ω_c and $1/\rho_c$:

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = qv_yB_z \qquad\qquad m\frac{\mathrm{d}v_y}{\mathrm{d}t} = qv_xB_z$$

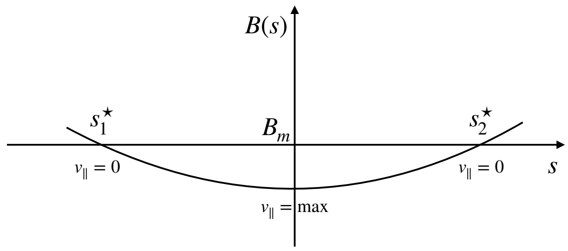
$$\ddot{x} + \omega_c^2 x = 0 \qquad\qquad \ddot{y} + \omega_c^2 y = 0.$$

This describes a harmonic oscillator and we already did this example in the previous lecture:

$$I = \pi m \omega_c \rho_c^2 \sim \frac{1}{2}|q|\omega_c \rho_c^2 \sim \mu.$$

The magnetic moment is the adiabatic invariant for the gyromotion around the z-axis.

2.2 *The parallel motion between the two mirror points*



Now let us consider the particle's bouncing motion parallel to the z-axis. In the previous section we have shown that the force acting on the plasma particle in z-direction is

$$F_z = -\frac{\partial B_s}{\partial z}\mu,$$

and thus

$$-\mu\frac{\partial B}{\partial s} = m\dot{v}_\| = m v_\| \frac{\partial v_\|}{\partial s} = m\frac{1}{2}\frac{\mathrm{d}v_\|^2}{\mathrm{d}s},$$

or

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}mv_\|^2 + \mu B\right),$$

implying that

$$W = \mu B_m = \frac{1}{2}mv_\|^2 + \mu B.$$

is constant and B_m is the maximum magnetic field strength. From this follows that the particle moves in an effective potential $\mu B(s)$ and

$$v_\|(s) = \pm\sqrt{\frac{2\mu}{m}(B_m - B(s))}.$$

Using the definition for the adiabatic invariant we get

$$I = m\oint v_\| \,\mathrm{d}s = \sqrt{2\mu m}\oint \sqrt{B_m - B}\,\mathrm{d}s$$

This defines the so-called *second adiabatic invariant* J, which is associated with the periodic bouncing motion

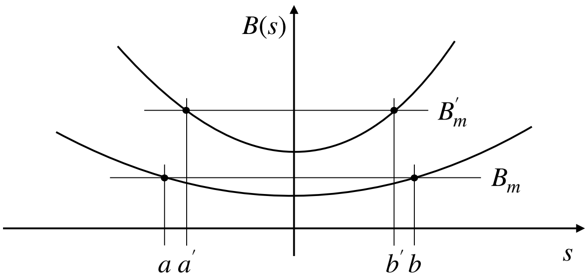
$$\boxed{J = \sqrt{2\mu m}\int_a^b \sqrt{B_m - B}\,\mathrm{d}s.} \tag{9}$$

As an application let us consider the case when the field strength of a magnetic bottle slowly changes, i.e.

$$B(s) \xrightarrow{\text{slowly}} B^{'}(s),$$

which implies that the energy is conserved. However, the second adiabatic invariant is conserved, i.e.

$$\sqrt{2\mu m}\int_a^b \sqrt{B_m - B(s)}\,\mathrm{d}s = \sqrt{2\mu m}\int_{a^{'}}^{b^{'}} \sqrt{B_m^{'} - B^{'}(s)}\,\mathrm{d}s.$$



2.3 *Fields with a pronounced axial symmetry*

Plasma particles immersed in an axially symmetric magnetic field will drift around the field axis in closed orbits. Remember that

$$\begin{aligned}
\mathbf{v}_E &= \frac{\mathbf{E}\times\mathbf{B}}{B^2} && \mathbf{v}_E \neq f(q,T)\\
\mathbf{v}_G &= \frac{T_\perp}{qB}\left[\frac{\hat{\mathbf{B}}\times\nabla\mathbf{B}}{B}\right] && \mathbf{v}_G = f(q,T)\\
\mathbf{v}_c &= \frac{2T_\parallel}{qB}\left[\frac{\hat{\mathbf{B}}\times\hat{\mathbf{R}}_c}{R_c}\right] && \mathbf{v}_G = f(q,T).
\end{aligned}$$

Consider an orbit close to the symmetry axis,

$$\int_C \mathbf{E}\cdot\mathrm{d}l = -\int_S \frac{\partial\mathbf{B}}{\partial t}\,\mathrm{d}\mathbf{A},$$

where we perform the integration along the drift contour

$$2\pi R E = -\pi R^2\frac{\mathrm{d}B}{\mathrm{d}t},$$

or

$$E = -\frac{R}{2}\frac{\mathrm{d}B}{\mathrm{d}t}.$$

E is the azimuthal electric field, which leads to a radial $\mathbf{E}\times\mathbf{B}$ drift

$$v_E = \frac{E}{B} = -\frac{R}{2B}\frac{\mathrm{d}B}{\mathrm{d}t} \stackrel{!}{=} \frac{R}{t},$$

or

$$2\frac{\mathrm{d}R}{R} = -\frac{\mathrm{d}B}{B}.$$

From this we obtain the *third adiabatic invariant*

$$\boxed{\Phi_B = \pi R^2 B,} \tag{10}$$

which is in fact the magnetic flux.